

Asymptotic Conformal Yano–Killing Tensors for Schwarzschild Metric

Jacek Jezierski

Department of Mathematical Methods in Physics,
University of Warsaw, ul. Hoża 74, 00-682 Warsaw, Poland

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Abstract

The *asymptotic conformal Yano–Killing tensor* proposed in [1] is analyzed for Schwarzschild metric and tensor equations defining this object are given. The result shows that the Schwarzschild metric (and other metrics which are asymptotically “Schwarzschildian” up to $O(1/r^2)$ at spatial infinity) is among the metrics fulfilling stronger asymptotic conditions and supertranslations ambiguities disappear. It is also clear from the result that 14 asymptotic gravitational charges are well defined on the “Schwarzschildian” background.

1 Introduction

We have proposed in [1] the charged solutions of spin-2 equations. The new charges result in a natural way from a geometric formulation of the “Gauss law” for the gravitational charges, defined in terms of the Riemann tensor (equation 5). It leads to the notion of the *conformal Yano–Killing tensor*. A conformal Yano–Killing (CYK) equation (2) possesses twenty-dimensional space of solutions for flat Minkowski metric in four-dimensional spacetime. This can be easily seen from our analysis when we pass to the limit with mass parameter m ($m \rightarrow 0$).

A natural application of the construction of CYK tensor to the description of asymptotically flat spacetimes was proposed in [1]. It allows us to define an asymptotic charge at spatial infinity without supertranslation ambiguities. The existence or nonexistence of the corresponding asymptotic CYK tensors can be chosen as a criterion for classification of asymptotically flat spacetimes. We show in this article that Schwarzschild metric is an example of a nice asymptotically flat spacetime from this point of view. It possesses a full set of 14 asymptotic CYK tensors.

2 Conformal Yano–Killing tensors and asymptotically flat spacetimes

Let $Q_{\mu\nu}$ be an antisymmetric tensor field fulfilling the following condition introduced by Penrose (see [2] and [3]):

$$Q_{\lambda(\kappa;\sigma)} - Q_{\kappa(\lambda;\sigma)} + \eta_{\sigma[\lambda} Q_{\kappa]}{}^\delta{}_{;\delta} = 0 \quad (1)$$

It is easy to check that equation (1) is equivalent to the following one:

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} - \frac{2}{3} \left(g_{\sigma\lambda} Q^\nu{}_{\kappa;\nu} + g_{\kappa(\lambda} Q_{\sigma)}{}^\mu{}_{;\mu} \right) = 0 \quad (2)$$

The tensor fulfilling the equation (1) or (2) we proposed to call the *conformal Yano Killing tensor* (or simply CYK). The CYK tensor is a natural “conformal” generalization of the Yano tensor.

Consider an asymptotically flat spacetime (at spatial infinity), fulfilling the Einstein equations. Suppose, moreover, that the energy–momentum tensor of the matter vanishes around spatial infinity (“sources of compact support”). Let us analyze, for simplicity, this situation in terms of an asymptotically flat coordinate system. We suppose that there exists an (asymptotically Minkowskian) coordinate system (x^μ) :

$$g_{\mu\nu} - \eta_{\mu\nu} = O(r^{-b}) \quad g_{\mu\nu,\lambda} = O(r^{-b-1})$$

where $r := \sum_{k=1}^3 (x^k)^2$ and typically $b = 1$ (but $1 \geq b > \frac{1}{2}$ is also possible).

For a general asymptotically flat metric we cannot expect that the equations (1) and (2) admit any solution. Instead, we assume that the left-hand side of (2):

$$\mathcal{Q}_{\lambda\kappa\sigma} := Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} - \frac{2}{3} \left(g_{\sigma\lambda} Q^\nu{}_{\kappa;\nu} + g_{\kappa(\lambda} Q_{\sigma)}{}^\mu{}_{;\mu} \right) \quad (3)$$

has certain asymptotic behaviour at spatial infinity

$$\mathcal{Q}_{\mu\nu\lambda} = O(r^{-c}) \quad (4)$$

On the other hand, suppose that $Q_{\mu\nu}$ behaves asymptotically as follows:

$$Q_{\mu\nu} = O(r^a) \quad Q_{\lambda\kappa,\sigma} = O(r^{a-1})$$

Moreover, suppose that the Riemann tensor $R_{\mu\nu\kappa\lambda}$ behaves asymptotically as follows:

$$R_{\mu\nu\kappa\lambda} = O(r^{-b-1-d})$$

It can be easily checked (see e.g. [3]) that the vacuum Einstein equations imply the following equality:

$$\nabla_\lambda \left({}^*R^{\mu\lambda}{}_{\alpha\beta} Q^{\alpha\beta} \right) = \frac{1}{3} R^{\mu\lambda\alpha\beta} Q_{\alpha\beta\lambda} \quad (5)$$

The left-hand side of (5) defines an asymptotic charge provided that the right-hand side vanishes sufficiently fast at infinity. It is easy to check that, for this purpose, the exponents b, c, d have to fulfill the inequality $b + c + d > 2$. In typical situation when $b = d = 1$, the above inequality simply means that $c > 0$. In this case a weaker condition is also possible (for example $Q_{\mu\nu\lambda} = O((\ln r)^{-1-\epsilon})$ with $\epsilon > 0$). Moreover, when $Q_{\mu\nu\lambda}$ vanishes the formula (5) gives “pure” charge (not only asymptotic).

Let us define an *asymptotic conformal Yano-Killing tensor* (ACYK) as an antisymmetric tensor $Q_{\mu\nu}$ such that $Q_{\mu\nu\lambda} \rightarrow 0$ at spatial infinity. For constructing the ACYK tensor we can begin with the solutions of (1) in flat Minkowski space. Asymptotic behaviour at infinity of these flat solutions explain why we expect for any ACYK tensor the following behaviour:

$$Q_{\mu\nu} = {}^{(2)}Q_{\mu\nu} + {}^{(1)}Q_{\mu\nu} + {}^{(0)}Q_{\mu\nu}$$

where ${}^{(2)}Q_{\mu\nu} = O(r^2)$, ${}^{(1)}Q_{\mu\nu} = O(r)$ and ${}^{(0)}Q_{\mu\nu} = O(r^{1-c})$.

It is easy to verify that $c \geq b + 1 - a$ and if $b = 1$ than for $a = 2$ we have $c \geq 0$. This means that in a general situation there are no solutions of (4) with nontrivial ${}^{(2)}Q_{\mu\nu}$ and $c > 0$. This is the origin of the difficulties with the definition of the angular momentum. On the other hand it is easy to check that the energy-momentum four-vector and the dual one are well defined ($a = c = 1$) and the condition $b + d > 1$ can be easily fulfilled (typically $b = d = 1$).

We proposed in [1] a new, stronger definition of the asymptotic flatness. The definition is motivated by the above discussion.

Suppose that there exists a coordinate system (x^μ) such that:

$$g_{\mu\nu} - \eta_{\mu\nu} = O(r^{-1})$$

$$\Gamma^\kappa{}_{\mu\nu} = O(r^{-2})$$

$$R_{\mu\nu\kappa\lambda} = O(r^{-3})$$

In the space of ACYK tensors fulfilling the asymptotic condition

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} - \frac{2}{3} \left(g_{\sigma\lambda} Q^\nu{}_{\kappa;\nu} + g_{\kappa(\lambda} Q_{\sigma)}{}^\mu{}_{;\mu} \right) = Q_{\lambda\kappa\sigma} = O(r^{-1}) \quad (6)$$

we define the following equivalence relation:

$$Q_{\mu\nu} \equiv Q'_{\mu\nu} \iff Q_{\mu\nu} - Q'_{\mu\nu} = O(1) \quad (7)$$

for $r \rightarrow \infty$. We assume that the space of equivalence classes defined by (6) and (7) has a finite dimension D as a vector space. The maximal dimension $D = 14$ correspond to the situation where there are no supertranslation problems in the definition of an angular momentum. In the case of spacetimes for which $D < 14$ the lack of certain ACYK tensor means that the corresponding charge is not well defined.

3 Conformal Yano–Killing tensors for Schwarzschild metric

For our purposes we need to specify the formula (2) to the special case of the Schwarzschild metric $g_{\mu\nu}$:

$$g_{\mu\nu} dx^\mu dx^\nu = - \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \quad (8)$$

We use radial coordinates: $x^3 = r$, $x^1 = \theta$, $x^2 = \varphi$. Moreover, $t = x^0$ denotes the time coordinate. We consider only part of the Schwarzschild spacetime far away from the horizon, $r \gg m$.

We use the following convention for indices: greek indices μ, ν, \dots run from 0 to 3; k, l, \dots are spatial coordinates and run from 1 to 3; A, B, \dots are spherical angles (θ, φ) on a two-dimensional sphere $S(r) := \{r = x^3 = \text{const.}\}$ and run from 1 to 2. Moreover, let η_{AB} denotes two-dimensional metric on $S(r)$.

Let $v := 1 - \frac{2m}{r}$. There are following non-vanishing Christoffel symbols for the metric (8):

$$\Gamma^3_{AB} = -\frac{v}{r}\eta_{AB}; \quad \Gamma^A_{3B} = \frac{1}{r}\delta^A_B; \quad \Gamma^3_{00} = \frac{mv}{r^2}; \quad \Gamma^0_{30} = \frac{m}{vr^2}; \quad \Gamma^A_{BC}$$

where δ^A_B is a Kronecker's symbol and Γ^A_{BC} are the same as for standard unit sphere $S(1)$.

For Schwarzschild metric (8) we obtain the following components of the equation (3):

$$Q_{003} = Q_{30;0} - \frac{1}{3}g_{00}Q_3{}^\nu{}_{;\nu} = \frac{2}{3}\dot{Q}_{30} + \frac{v}{3}Q_{3A||B}\eta^{AB} \quad (9)$$

$$Q_{00A} = Q_{A0;0} - \frac{1}{3}g_{00}Q_A{}^\nu{}_{;\nu} = \frac{2}{3}\dot{Q}_{A0} + \frac{2mv}{3r^2}Q_{3A} + \frac{v}{3}Q_{AB||C}\eta^{BC} + \frac{v^2}{3}Q_{A3,3} \quad (10)$$

$$Q_{033} = Q_{03;3} - \frac{1}{3}g_{33}Q_0{}^\nu{}_{;\nu} = \frac{2}{3}Q_{03,3} - \frac{2m}{3vr^2}Q_{03} - \frac{2}{3r}Q_{03} - \frac{1}{3v}Q_{0A||B}\eta^{AB} \quad (11)$$

$$\mathcal{Q}_{03A} = Q_{03;A} + Q_{A3;0} = Q_{03,A} - \frac{1}{r}Q_{0A} + \dot{Q}_{A3} + \frac{m}{vr^2}Q_{0A} \quad (12)$$

$$\begin{aligned} \mathcal{Q}_{0AB} &= Q_{0A;B} + Q_{BA;0} - \frac{1}{3}\eta_{AB}Q_0{}^\nu{}_{;\nu} = Q_{0A||B} - \frac{1}{3}\eta^{CD}Q_{0C||D}\eta_{AB} - \dot{Q}_{AB} + \\ &+ \frac{1}{3}\eta_{AB}\left(\frac{v}{r}Q_{03} + \frac{m}{r^2}Q_{03} - vQ_{03,3}\right) \end{aligned} \quad (13)$$

$$\mathcal{Q}_{30A} = Q_{30;A} + Q_{A0;3} = Q_{30,A} + \frac{2}{r}Q_{0A} + Q_{A0,3} + \frac{m}{vr^2}Q_{0A} \quad (14)$$

$$\begin{aligned} \mathcal{Q}_{A0B} &= Q_{A0;B} + Q_{B0;A} + \frac{2}{3}\eta_{AB}Q_0{}^\nu{}_{;\nu} = Q_{A0||B} + Q_{B0||A} + \frac{2}{3}\eta^{CD}Q_{0C||D}\eta_{AB} + \\ &- \frac{2}{3}\eta_{AB}\left(\frac{v}{r}Q_{03} + \frac{m}{r^2}Q_{03} - vQ_{03,3}\right) \end{aligned} \quad (15)$$

$$\begin{aligned} \mathcal{Q}_{33A} &= Q_{A3;3} - \frac{1}{3}g_{33}Q_A{}^\nu{}_{;\nu} = \frac{2}{3}Q_{A3,3} + \frac{m}{3vr^2}Q_{3A} + \frac{1}{r}Q_{3A} - \frac{1}{3v^2}\dot{Q}_{0A} + \\ &- \frac{1}{3v}Q_A{}^B{}_{||B} \end{aligned} \quad (16)$$

$$\begin{aligned} \mathcal{Q}_{3AB} &= Q_{3A;B} + Q_{BA;3} - \frac{1}{3}\eta_{AB}Q_3{}^\nu{}_{;\nu} = Q_{3A||B} - \frac{1}{3}\eta_{AB}\left(Q_3{}^C{}_{||C} + v^{-1}\dot{Q}_{03}\right) + \\ &+ Q_{BA,3} + \frac{3}{r}Q_{AB} \end{aligned} \quad (17)$$

$$\begin{aligned} \mathcal{Q}_{A3B} &= Q_{A3;B} + Q_{B3;A} + \frac{2}{3}\eta_{AB}Q_3{}^\nu{}_{;\nu} = Q_{A3||B} + Q_{B3||A} + \frac{2}{3}\eta_{AB}Q_3{}^C{}_{||C} + \\ &+ \frac{2}{3v}\eta_{AB}\dot{Q}_{03} \end{aligned} \quad (18)$$

$$\begin{aligned} \mathcal{Q}_{ABC} &= Q_{AB;C} + Q_{CB;A} - \frac{2}{3}\left(\eta_{AC}Q^{\nu}{}_{B;\nu} + \eta_{B(A}Q_{C)}{}^\nu{}_{;\nu}\right) = Q_{AB||C} + Q_{CB||A} + \\ &+ \frac{2}{3}\eta_{AC}Q_B{}^D{}_{||D} - \frac{2}{3}\eta_{B(A}Q_{C)}{}^D{}_{||D} + \frac{2v}{r}\left(\eta_{AC}Q_{3B} + \eta_{B(A}Q_{C)3}\right) + \frac{2}{3v}\eta_{AC}\dot{Q}_{0B} + \\ &+ \frac{2}{3v}\eta_{B(A}\dot{Q}_{C)0} + \frac{2}{3}\eta_{AC}\left(vQ_{B3,3} + \frac{m}{r^2}Q_{B3}\right) - \frac{2}{3}\eta_{B(A}\left(vQ_{C)3,3} + \frac{m}{r^2}Q_{C)3}\right) \end{aligned} \quad (19)$$

where “;” denotes four-dimensional covariant derivative with respect to the Schwarzschild metric $g_{\mu\nu}$, by dot we have denoted as usual time derivative and symbol “||” denotes two-dimensional covariant derivative with respect to the two-metric η_{AB} .

From (15) and (11) we get:

$$\mathcal{Q}_{A0B} - v\mathcal{Q}_{033} = Q_{0C||D}\eta^{CD}\eta_{AB} - Q_{0A||B} - Q_{0B||A} \quad (20)$$

Similarly from (18) and (9) we have

$$\mathcal{Q}_{A3B} + v^{-1}\mathcal{Q}_{003} = Q_{3C||D}\eta^{CD}\eta_{AB} - Q_{3A||B} - Q_{3B||A} \quad (21)$$

Let \mathbf{a} denotes the two-dimensional Laplace–Beltrami operator on a unit sphere $S(1)$ and on each sphere $S(r)$ we denote by ε^{AB} the Levi–Civita antisymmetric tensor such that $r^2 \sin \theta \varepsilon^{12} = 1$. For a function f on $S(r)$ we denote by ${}^m f$ its monopole part, by ${}^d f$ its dipole part and we denote by ${}^w f$ the remainder, which will be called the “radiation part”. The equations (20) and (21) show that only mono–dipole part of Q_{0A} and Q_{3A} can have nontrivial asymptotic behaviour, the higher poles have to vanish at spatial infinity. More precisely:

$$\begin{aligned} {}^w Q_{0A||B}\eta^{AB} &= O\left(\frac{1}{r}\right), & {}^w Q_{0A||B}\varepsilon^{AB} &= O\left(\frac{1}{r}\right) \\ {}^w Q_{3A||B}\eta^{AB} &= O\left(\frac{1}{r}\right), & {}^w Q_{3A||B}\varepsilon^{AB} &= O\left(\frac{1}{r}\right) \end{aligned}$$

On each sphere $S(r)$ the full information about tensor $Q_{\mu\nu}$ is encoded in the following 6 scalar functions: Q_{03} , $Q_{0A||B}\eta^{AB}$, $Q_{3A||B}\eta^{AB}$, $Q_{0A||B}\varepsilon^{AB}$, $Q_{0A||B}\varepsilon^{AB}$, q where $q := \frac{1}{2}Q_{AB}\varepsilon^{AB}$. The full set of the above components splits, in a natural way, into two sets: $\{ Q_{03}, Q_{0A||B}\eta^{AB}, Q_{3A||B}\eta^{AB} \}$ and $\{ Q_{0A||B}\varepsilon^{AB}, Q_{0A||B}\varepsilon^{AB}, q \}$. Let us notice that the equations (9–19) can be also splitted in the same way. The “dynamics” of the first set can be described by the following relations:

$$3\mathcal{Q}_{003} = 3v\mathcal{Q}_{3AB}\eta^{AB} = -2\dot{Q}_{03} + vQ_{3A||B}\eta^{AB} \quad (22)$$

$$3r\mathcal{Q}_{00}{}^A{}_{||A} = -2r\dot{Q}_0{}^A{}_{||A} + \frac{2mv}{r}Q_3{}^A{}_{||A} - v^2rQ_{3A||B,3}\eta^{AB} \quad (23)$$

$$r\mathcal{Q}_{03}{}^A{}_{||A} = -r\dot{Q}_3{}^A{}_{||A} + rQ_{03}{}^{||A}{}_{||A} - Q_0{}^A{}_{||A} + \frac{m}{vr}Q_0{}^A{}_{||A} \quad (24)$$

$$3\mathcal{Q}_{033} = -\frac{3}{v}\mathcal{Q}_{0AB}\eta^{AB} = \frac{3}{2v}\mathcal{Q}_{A0B}\eta^{AB} = 2Q_{03,3} - \frac{2m}{vr^2}Q_{03} - \frac{2}{r}Q_{03} - \frac{1}{v}Q_0{}^A{}_{||A} \quad (25)$$

$$r\mathcal{Q}_{30}{}^A{}_{||A} = -r\left(Q_0{}^A{}_{||A}\right)_{,3} - rQ_{03}{}^{||A}{}_{||A} + \frac{m}{vr}Q_0{}^A{}_{||A} \quad (26)$$

$$\frac{r}{v^2}\mathcal{Q}_{00}{}^A{}_{||A} - 2r\mathcal{Q}_{33}{}^A{}_{||A} = r\left(Q_3{}^A{}_{||A}\right)_{,3} \quad (27)$$

There are more equations but they are linearly dependent, for example:

$$\begin{aligned}
3v^{-1}r\mathcal{Q}_{00}{}^A{}_{||A} - 3vr\mathcal{Q}_{33}{}^A{}_{||A} &= 3r\mathcal{Q}^{ABC}{}_{||A}\eta_{BC} = -\frac{r}{v}\dot{Q}_0{}^A{}_{||A} - vQ_3{}^A{}_{||A} + \\
&+ \frac{m}{r}Q_3{}^A{}_{||A} + vr\left(Q_3{}^A{}_{||A}\right)_{,3}
\end{aligned} \tag{28}$$

From (22) and (25) we obtain the following exact solution for the monopole part of Q_{03} :

$${}^mQ_{03} = rv^{-\frac{1}{2}}$$

The dipole part ${}^dQ_{03}$, ${}^dQ_{0A||B}\eta^{AB}$, ${}^dQ_{3A||B}\eta^{AB}$ can be obtained from eq. (9-27) only asymptotically:

$$\begin{aligned}
2{}^dQ_{03} &= K\left(r^2 - mr - t^2\right) + Pt \\
r{}^dQ_0{}^A{}_{||A} &= K\left(r^2 - 3mr + t^2\right) - Pt \\
r{}^dQ_3{}^A{}_{||A} &= Kt(5m - 2r) + Pr
\end{aligned} \tag{29}$$

where K , P are dipole functions on a unit sphere and solution is unique up to $O(1)$.

Similarly for $Q_{3A||B}\varepsilon^{AB}$, $Q_{0A||B}\varepsilon^{AB}$ and q we have the following relations:

$$r\mathcal{Q}_{03A||B}\varepsilon^{AB} = -r\dot{Q}_{3A||B}\varepsilon^{AB} - Q_{0A||B}\varepsilon^{AB} + \frac{m}{vr}Q_{0A||B}\varepsilon^{AB} \tag{30}$$

$$\mathcal{Q}_{0AB}\varepsilon^{AB} = Q_{0A||B}\varepsilon^{AB} - 2\dot{q} \tag{31}$$

$$\begin{aligned}
\frac{r}{v^2}\mathcal{Q}_{00A||B}\varepsilon^{AB} + r\mathcal{Q}_{33A||B}\varepsilon^{AB} &= -\frac{r}{v^2}\dot{Q}_{0A||B}\varepsilon^{AB} - \left(rQ_{3A||B}\varepsilon^{AB}\right)_{,3} + \\
&+ \frac{m}{vr}Q_{3A||B}\varepsilon^{AB}
\end{aligned} \tag{32}$$

$$\frac{r}{v^2}\mathcal{Q}_{00A||B}\varepsilon^{AB} - 2r\mathcal{Q}_{33A||B}\varepsilon^{AB} = r\left(Q_{3A||B}\varepsilon^{AB}\right)_{,3} + \frac{1}{rv}\mathbf{a}q \tag{33}$$

$$\mathcal{Q}_{3AB}\varepsilon^{AB} = Q_{3A||B}\varepsilon^{AB} - 2q_{,3} + \frac{2}{r}q \tag{34}$$

$$r\mathcal{Q}_{30A||B}\varepsilon^{AB} = -r\left(Q_{0A||B}\varepsilon^{AB}\right)_{,3} + \frac{m}{vr}Q_{0A||B}\varepsilon^{AB} \tag{35}$$

We obtain an exact monopole solution from (31) and (34):

$${}^mq = r$$

An asymptotic solution of (30–35) for ${}^dQ_{3A||B}\varepsilon^{AB}$, ${}^dQ_{0A||B}\varepsilon^{AB}$ and dq has the following form (up to $O(1)$):

$$\begin{aligned} 2{}^dq &= J(r^2 - 2mr - t^2) + Bt \\ r{}^dQ_{3A||B}\varepsilon^{AB} &= J(r^2 + t^2) - Bt \\ r{}^dQ_{0A||B}\varepsilon^{AB} &= Jt(9m - 2r) + Br \end{aligned} \tag{36}$$

where again J and B are dipoles on a unit sphere:

$$(\mathbf{a} + 2)J = (\mathbf{a} + 2)B = J_{,3} = B_{,3} = \dot{J} = \dot{B} = 0$$

It is easy to check that for $m = 0$ the dipole solutions (29) and (36) are exact. This way we get 14 dimensional space of CYK tensors for flat Minkowski space (12 dipoles plus 2 monopoles). In this case we have also 6 more constant CYK tensors fullfilling equation (2) so the total space of CYK tensors for flat Minkowski space is 20 dimensional [1], [2].

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